Second sound in an anisotropic quasiparticle system of superfluid ^{4}He

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The phase and group velocities of second sound modes in superfluid helium are obtained for arbitrary values of the relative velocity of the normal and superfluid components. We show that the phase and group velocities of second sound, in general, depend on the angle between the wave vector and the relative velocity between the normal and superfluid components **w**. We have found the relationship between the amplitudes of the oscillating variables that describe second sound. In the general case, the normal fluid not only has a velocity component parallel to the wave vector, but also a transverse velocity component. The general expressions for the velocities and amplitudes are analyzed when the normal fluid is only due to phonons. We find that there is a certain value of **w** which makes the second-sound wave stationary in the laboratory frame. We show that the amplitude of the temperature oscillation, in a second-sound wave in an anisotropic phonon system, can be zero under some conditions.

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I. INTRODUCTION

In superfluid ^{4}He, it is possible to create pulses of thermal excitations which have high values of the relative velocity between the normal fluid and the superfluid.[1] Such pulses can be injected into liquid helium, which is so cold that it has negligible thermal excitations, by a heater. Another way to create a large relative velocity is to make the helium move through narrow channels which lock the normal fluid. In both these systems, the velocity of the normal fluid, or the superfluid, can be very close to the critical velocity. Such systems have been investigated both theoretically[2–5] and experimentally.[6–10]

Both first and second-sound modes in superfluid helium have been investigated over a long period (see, for example, Ref. 11 and references therein). But almost all these considerations were limited to zero or small values of the relative velocity $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$, where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal and superfluid velocities. As we now have the opportunity to experimentally investigate the propagation of sound modes with high values of $w=|\mathbf{w}|$, we need to extend the existing theory of sound propagation in superfluid helium to high values of the relative velocity, which can be close to the critical velocity of a phonon system. The first theoretical step was made in Ref. 12, where the velocity of first and second sound was found for small w, i.e., in an approximation which is linear in w. Also, the thermal-expansion coefficient of liquid helium was assumed to be zero.

An attempt to solve the problem of second-sound propagation for arbitrary values of w was made in Ref. 13. In that paper, liquid helium is taken to be incompressible, with zero oscillation of the total momentum. The pressure was then found from the expression for the momentum flow tensor and the momentum conservation law, without considering the other equations of the system. But in fact assuming incompressibility of helium and zero oscillations of the total momentum in second sound imply that we should omit the momentum conservation law from the set of equations that describe the system. Thus, the dispersion relation of second sound obtained in Ref. 13 is incorrect.

Some preliminary results on the dispersion relation for second sound at arbitrary values of w, assuming incompressibility of helium, were reported at the conference in Ref. 14. The relationship between the amplitudes of the variables describing first sound, at arbitrary values of the relative velocity w and when the thermal excitation contribution is small, were obtained in Ref. 15. The existence of the transverse mode, and the general relations between the amplitudes in this mode for arbitrary value of w, were established in Ref. 16, where we also discussed the possibility of experimentally detecting the transverse mode in phonon pulses in superfluid helium.

It should be noted that the experiments on scattering and propagating phonon pulses in superfluid helium do not show any scattering from vortex lines. So it would appear that high values of the relative velocity in phonon pulses do not create enough vortices, either within the propagating phonon pulse or outside it in superfluid helium, to be a problem. We also see from experiments (see, for example, Refs. 6, 9, and 10 and the references cited therein) that there exist phonon systems (phonon pulses) that have a velocity equal to the sound velocity relative to the superfluid. Such phonon systems do not include either rotons or vortices. Other experiments (see, for example, Ref. 21) show that in these strongly anisotropic phonon systems, at low pressure, phonons strongly interact and this interaction leads to a quasiequilibrium state. The phonon systems described above have also been intensively studied theoretically. In particular, it was shown theoretically using general relations that quasiparticle systems, with large values of w close to the sound velocity, are thermodynamically stable,[2] and that they possess unique thermodynamic properties.[4,5] These points are discussed in detail at the beginning of Sec. V.

In this paper we find the phase and group velocities for second sound at arbitrary values of **w**, using the hydrodynamic equations for superfluid helium, with the condition that the normal density ρ_n is small compared to the helium density ρ. We show that the phase and group velocities of second sound in general depend on the angle between the wave vector and $\mathbf{w}$. The phase and group velocities can be characterized by three parameters: the longitudinal $V_{\parallel}$, perpendicular $V_{\perp}$, and drift V_d velocities.

We find the relationships between the amplitudes of the oscillating hydrodynamic variables in a second-sound wave. The general expressions for the amplitudes are analyzed when the normal fluid is only composed of phonons. Then we find that the amplitude of the temperature oscillation, in a second-sound wave in an anisotropic phonon system, can be zero under some conditions. Furthermore, there is a value of w such that the second-sound wave is stationary in the laboratory frame. This condition is connected to wave blocking in classical systems (see, e.g., Ref. 17 and horizons in white hole analogs[18,19]).

II. SYSTEM OF HYDRODYNAMIC EQUATIONS

We are interested in describing the evolution of small deviations of various parameters in helium from their stationary values. The typical feature of strongly anisotropic systems is their high value of the relative velocity w, which can be comparable with the critical Landau velocity.[11] For phonon systems such as the phonon pulses created in superfluid liquid in experiments,[1,6,9,10] w turns out to be close to the first sound velocity. For phonon-roton systems, the maximum value of w is determined by the stability conditions,[2,4] and at sufficiently low temperatures, can be close to the phase velocity of rotons. So, we will linearize the system of hydrodynamic equations for superfluid helium without supposing that the stationary values of superfluid and normal velocities are small, as was supposed before.[11,12,20]

For small deviations from the stationary values we find a solution, in the linear approximation, in the form of plane waves $\exp(i\mathbf{kr}-i\omega t)$, where $\mathbf{k}$ is the wave vector and ω is the angular frequency. From the well-known hydrodynamic equations for superfluid helium,[11,20] in the nondissipative approximation, we get the following system of linear equations for small deviations:

$$\tilde{\mathbf{v}}_s = \frac{\tilde{\mu}}{(u - v_{s\parallel})}\frac{\mathbf{k}}{k}, \tag{1}$$

$$(v_{n\parallel} - u)\tilde{S} + S\tilde{v}_{n\parallel} = 0, \tag{2}$$

$$(v_{s\parallel} - u)\tilde{\rho} + w_{\parallel}\tilde{\rho}_n + \rho_n\tilde{v}_{n\parallel} + \rho_s\tilde{v}_{s\parallel} = 0, \tag{3}$$

$$(v_{n\parallel} - u)\left[\tilde{\mathbf{w}} + \mathbf{w}\left(\frac{\tilde{\rho}_n}{\rho_n} - \frac{\tilde{S}}{S}\right)\right] + \left(\frac{S}{\rho_n}\tilde{T} + \mathbf{w}\tilde{\mathbf{v}}_n\right)\frac{\mathbf{k}}{k} = 0. \tag{4}$$

Here μ is the chemical potential of unit of mass of helium; S is entropy of unit of volume; T is absolute temperature in energy units. Small deviation of the values are marked with the symbol "tilde," and from now on, a symbol without a "tilde" means the constant value of that variable. Here $v_{s\parallel} = \mathbf{v}_s\mathbf{k}/k$, and similarly for $v_{n\parallel}$ and $w_{\parallel}$. The phase velocity of a mode is given by $u=\omega/k$. From the condition that superfluid motion is potential flow, i.e., $\mathbf{v}_s=\nabla\varphi$, where φ is the superfluid velocity potential [which is not included in the system of Eqs. (1)–(4)], it follows that the oscillation of superfluid velocity is always longitudinal, i.e., parallel to $\mathbf{k}$. This result is in agreement with Eq. (1) when $u\neq v_{s\parallel}$. So the mode $u=v_{s\parallel}$, which is possible from system of Eqs. (1)–(4), is prohibited by the additional condition of the existence of a superfluid velocity potential. At the same time, from Eq. (4) it follows that the oscillation of the relative velocity $\tilde{\mathbf{w}}$, as well as normal velocity $\tilde{\mathbf{v}}_n$, has in the general case both longitudinal and transverse components relative to vector $\mathbf{k}$ if the constant value of $\mathbf{w}$ is not equal to zero. In contrast, when $\mathbf{w}=0$, the isotropic case, we only have velocity components parallel to $\mathbf{k}$ so the normal fluid, as well as the superfluid, has a velocity potential.

The system of Eqs. (1)–(4) determines the relations between the amplitudes of the oscillating values in the five various modes since we have five independent equations for five variables. These five variables can be chosen as follows: $\tilde{v}_{s\parallel}$ and two projections of relative velocity $\tilde{\mathbf{w}}$ oscillations, which lie in the plane formed by the vectors $\mathbf{w}$ and $\mathbf{k}$, and two scalar values, pressure $\tilde{P}$ and temperature $\tilde{T}$. All other thermodynamic variables can be expressed by these variables, with the help of the thermodynamic equations

$$\tilde{S} = \left(\frac{\partial S}{\partial P}\right)_{T,w}\tilde{P} + \left(\frac{\partial S}{\partial T}\right)_{P,w}\tilde{T} + \left(\frac{\partial S}{\partial w^2/2}\right)_{P,T}\mathbf{w}\tilde{\mathbf{w}}, \tag{5}$$

$$\tilde{\rho}_n = \left(\frac{\partial \rho_n}{\partial P}\right)_{T,w}\tilde{P} + \left(\frac{\partial \rho_n}{\partial T}\right)_{P,w}\tilde{T} + \left(\frac{\partial \rho_n}{\partial w^2/2}\right)_{P,T}\mathbf{w}\tilde{\mathbf{w}}, \tag{6}$$

$$\tilde{\rho} = \left(\frac{\partial \rho}{\partial P}\right)_{T,w}\tilde{P} + \left(\frac{\partial \rho}{\partial T}\right)_{P,w}\tilde{T} + \left(\frac{\partial \rho}{\partial w^2/2}\right)_{P,T}\mathbf{w}\tilde{\mathbf{w}}, \tag{7}$$

and the thermodynamic relation for the chemical potential

$$\tilde{\mu} = \frac{1}{\rho}\tilde{P} - \frac{S}{\rho}\tilde{T} - \frac{\rho_n}{\rho}\mathbf{w}\tilde{\mathbf{w}}. \tag{8}$$

Among the nine thermodynamic derivatives in Eqs. (5)–(7) only six are independent because from Eq. (8) we have the thermodynamic equalities

$$\frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)_{P,w} = -\frac{\partial}{\partial P}\left(\frac{S}{\rho}\right)_{T,w}, \tag{9}$$

$$\frac{\partial}{\partial w^2/2}\left(\frac{S}{\rho}\right)_{P,T} = \frac{\partial}{\partial T}\left(\frac{\rho_n}{\rho}\right)_{P,w}, \tag{10}$$

$$\frac{\partial}{\partial w^2/2}\left(\frac{1}{\rho}\right)_{P,T} = -\frac{\partial}{\partial P}\left(\frac{\rho_n}{\rho}\right)_{T,w}. \tag{11}$$

The dispersion relation for one of the modes can be easily found from Eq. (4); At $u=v_{n\parallel}=\mathbf{v}_n\mathbf{k}/k$, instead of two independent equations in Eq. (4) we get just one equation. So, in superfluid helium there exists the mode $\omega=\mathbf{kv}_n$ which is the transverse mode.[16]

To analyze the remaining four modes, which correspond to the first and second sounds, it is convenient to choose the coordinate frame with axis x directed along the equilibrium value of the relative velocity $\mathbf{w}$, and axis y in the plane that

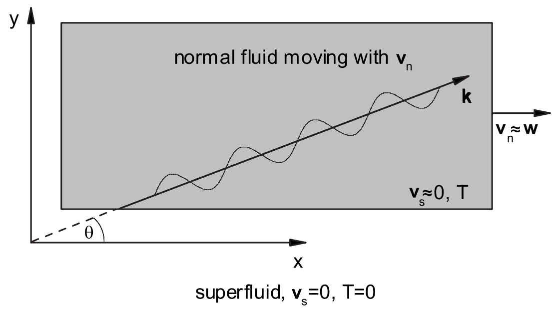

FIG. 1. A region of normal fluid is shown schematically moving through superfluid heluim. The coordinate frame is defined with the x axis directed along the vector of the relative velocity $\mathbf{w}=\mathbf{v_n}-\mathbf{v_s}$, and the y axis lying in the plane which is defined by vector $\mathbf{w}$ and wave vector $\mathbf{k}$. The angle θ is between vectors $\mathbf{k}$ and $\mathbf{w}$.

is determined by vector $\mathbf{w}$ and wave vector $\mathbf{k}$ with the condition $k_y > 0$ (see Fig. 1). If we take into account that Eqs. (5)–(8) are expressed only in terms of $\tilde{P}$ and $\tilde{T}$, and $\tilde{w}_x$, where $\tilde{w}_x$ is the x component of the oscillation of the relative velocity, we can simplify the system [Eqs. (1)–(4)]. To do this we take the yth component of Eq. (4) and using Eq. (1) for the oscillation of yth component we get

$$\tilde{w}_y = \frac{\sin\theta}{u - v_{n\parallel}}\left[\frac{S}{\rho_n}\tilde{T} + w\tilde{w}_x + \frac{w_\parallel}{u - v_{s\parallel}}\tilde{\mu}\right]. \quad (12)$$

Eliminating $\tilde{v}_{s\parallel}$, Eq. (1), and $\tilde{w}_y$, Eq. (12), from the remaining Eqs. (2) and (3) and the xth component of Eq. (4), we get the system of equations, which determine first and second sound

$$\tilde{w}_x + \cos\theta(u - v_{s\parallel})\frac{\tilde{\rho}}{\rho_s} + w\sin^2\theta\left(\frac{\tilde{\rho}_n}{\rho_n} - \frac{\tilde{S}}{S}\right) - w\cos^2\theta\frac{\tilde{\rho}_n}{\rho_s} - \cos\theta(u - v_{n\parallel})\frac{\rho}{\rho_s}\frac{\tilde{S}}{S} = 0, \quad (13)$$

$$\cos\theta\frac{S}{\rho_n}\tilde{T} - (u - v_{n\parallel})\tilde{w}_x + w\cos^2\theta(u - v_{n\parallel})\frac{\tilde{S}}{S} + w[u - v_{n\parallel} + w\cos\theta\sin^2\theta]\left(\frac{\tilde{S}}{S} - \frac{\tilde{\rho}_n}{\rho_n}\right) = 0, \quad (14)$$

$$\frac{1}{\rho}\tilde{P} - \frac{S}{\rho}\tilde{T} - \frac{\rho_n}{\rho}w\tilde{w}_x + w\cos\theta(u - v_{s\parallel})\frac{\tilde{\rho}_n}{\rho_s} + (u - v_{s\parallel})(u - v_{n\parallel})\frac{\rho_n}{\rho_s}\frac{\tilde{S}}{S} - (u - v_{s\parallel})^2\frac{\tilde{\rho}}{\rho_s} = 0. \quad (15)$$

The system of Eqs. (13)–(15), with relations (5)–(7), gives three equations for the three variables $\tilde{P}$, $\tilde{T}$, and $\tilde{w}_x$. The consistency condition for these equations gives the dispersion relations for first and second sounds. This relation is a quartic equation with four solutions and is explicitly presented in Ref. 15. In that paper the first sound mode, in anisotropic quasiparticle system of superfluid helium, was studied for the case when the contribution of thermal quasiparticles $\rho_n \ll \rho$ is small, which corresponds to experiments.[1,6,9,10] In this approximation, the velocity of first sound is $u = \pm c$, where $c^2 = (\partial P/\partial\rho)_{T=0}$. In this paper we will consider the second-sound mode in an anisotropic quasiparticle system of superfluid helium.

Here we consider the wave vector $\mathbf{k}$ is fixed, but the frequency ω (or phase velocity u) should be determined from the consistency conditions. Thus we may obtain modes with positive or negative frequencies. In such an approach, the sign of the frequency means nothing more than the direction of wave propagation. A positive frequency means the wave propagating from left to right (along $\mathbf{k}$), and a negative frequency means the wave propagating from right to left (against $\mathbf{k}$). Two solutions, which correspond to waves propagating in the opposite directions, also exist in classical acoustics.

III. PHASE AND GROUP VELOCITIES OF SECOND SOUND

We consider second sound when the density of thermal excitations is very small, i.e., $\rho_n/\rho \ll 1$. This is an important case as it can be experimentally attained.[1,6–10] We choose the coordinate frame in which the total equilibrium value of the full momentum of helium $\mathbf{j} = \mathbf{v}_n\rho_n + \mathbf{v}_s\rho_s = 0$. In this frame, the helium is motionless as a whole. This frame can always be achieved by a Galilean transformation. In this frame, and using the inequality $\rho_n/\rho \ll 1$, we get in this approximation

$$\mathbf{v}_n = \mathbf{w}, \quad \mathbf{v}_s = 0. \quad (16)$$

In all further results, we confine ourselves to the first nonvanishing term in the series of the small parameter ρ_n/ρ. Let us suppose that the pressure oscillation is proportional to the small density of thermal excitations in the normal fluid. In contrast, the relative oscillation of temperature and relative velocity depend, in general, on the zeroth order of $\rho_n/\rho \ll 1$. These statements will be justified later. From the equalities (9) and (11), we see that the coefficient of thermal expansion of helium and the density depend on the relative velocity w, and is determined only by the thermal excitations, and so is small in this approximation. Hence, the contributions of the second and the third terms in equality (7) are first order in $\rho_n/\rho \ll 1$, as well as the first term. Thus in this approximation it follows from the thermodynamic relation (7) that density oscillations also depends, to first order, on $\rho_n/\rho \ll 1$.

To obtain the relationships between the amplitudes of the oscillating values of temperature $\tilde{T}$ and the projection of the relative velocity $\tilde{w}_x$ onto $\mathbf{w}$, and to derive the dispersion relation of second sound in zeroth order of $\rho_n/\rho \ll 1$, we substitute relation (16) into Eqs. (13) and (14). Omitting the second and fourth terms in Eq. (13), as they are a higher order of smallness, and assuming that $\rho_s \approx \rho$ we get the following system of equations:

$$\tilde{w}_x + w\sin^2\theta\left(\frac{\tilde{\rho}_n}{\rho_n} - \frac{\tilde{S}}{S}\right) - (u - w\cos\theta)\cos\theta\frac{\tilde{S}}{S} = 0, \quad (17)$$

$$\cos\theta\frac{S}{\rho_n}\tilde{T}-(u-w\cos\theta)\tilde{w}_x+w\cos^2\theta(u-w\cos\theta)\frac{\tilde{S}}{S}$$
$$+w(u-w\cos^3\theta)\left(\frac{\tilde{S}}{S}-\frac{\tilde{\rho}_n}{\rho_n}\right)=0. \quad (18)$$

These equations should be completed by the thermodynamic relations for entropy [Eq. (5)] and normal density [Eq. (6)], in which we should omit the first terms which involve pressure, as they are determined by the small contribution of the thermal excitations to the second-sound oscillations. The condition for a nontrivial solution of this system of two equations, for two variables ($\tilde{T}$ and $\tilde{w}_x$), gives the dispersion equation to second order with respect to the phase velocity u. This is because Eqs. (17) and (18) are first order in u and the thermodynamic relations (5) and (6) do not include u. This equation describes two modes of second sound in helium at arbitrary values of $\mathbf{w}$. The solution of this equation gives the dispersion relation of second sound in helium,

$$u=\frac{\omega}{k}=V_d\cos\theta\pm\sqrt{V_\parallel^2\cos^2\theta+V_\perp^2\sin^2\theta}, \quad (19)$$

where we use the following notation:

$$V_d=w(1+\alpha),\quad \alpha=\frac{1-\partial\ln\rho_n/\partial\ln S}{1+\Gamma w^2}, \quad (20)$$

$$V_\parallel^2=\alpha^2w^2+u_2^2,\quad u_2^2=\frac{c_2^2}{1+\Gamma w^2}, \quad (21)$$

$$V_\perp^2=(\alpha+\beta)w^2+u_2^2,$$

$$\beta=\frac{c_2^2\,\partial\ln\rho_n/\partial\ln(w^2/2)-\partial\ln\rho_n/\partial\ln S}{1+\Gamma w^2}, \quad (22)$$

$$c_2=\pm\sqrt{\frac{S^2}{\rho_n}\left(\frac{\partial S}{\partial T}\right)^{-1}}, \quad (23)$$

$$\Gamma=\left(\frac{\partial\ln\rho_n}{\partial w^2/2}\right)-\frac{\rho_n}{S}\left(\frac{\partial\ln\rho_n}{\partial\ln S}\right)\left(\frac{\partial\ln\rho_n}{\partial T}\right), \quad (24)$$

where $\partial\ln\rho_n/\partial\ln S=(\partial\ln\rho_n/\partial T)(\partial\ln S/\partial T)^{-1}$. In Eqs. (19) and (23) the signs "plus" and "minus" correspond to the two solutions for waves with different phase velocities (see the end of Sec. II).

It follows from expression (19) that the phase velocity of second sound in helium, at $\mathbf{w}\neq 0$, is anisotropic. This dependence can be described by three characteristic velocities: V_d, $V_\parallel$, and $V_\perp$. To clarify the physical meaning of these velocities, we calculate, using expression (19), the phase velocity u for the limiting cases $\theta=0$ and $\theta=\pi/2$,

$$u(\theta=0)=V_d\pm V_\parallel,\quad u(\theta=\pi/2)=\pm V_\perp. \quad (25)$$

These equations show that $V_\parallel$ and $V_\perp$ are associated with the longitudinal and transverse phase velocities and V_d is the velocity of drift along the direction of $\mathbf{w}$.

In the linear approximation, with respect to w, the Eqs. (20)–(24) give

$$V_d=w(2-\partial\ln\rho_n/\partial\ln S)+O(w^3),\quad w\to 0, \quad (26)$$

and

$$V_\parallel^2=V_\perp^2=c_2^2+O(w^2),\quad w\to 0. \quad (27)$$

From the last equation it follows that both longitudinal and transverse velocities coincide with the usual isotropic velocity of second sound in the linear approximation with respect to w. The drift velocity is proportional to $\mathbf{w}$, but does not exactly equal it. The relations (26) and (27), after their substitution into Eq. (19), give the result first derived in Ref. 12. In that paper the dispersion relation of second sound was obtained only to first order in the velocity w, and the contribution of thermal expansion was neglected.

Let us note that we have derived the second-sound dispersion law [Eq. (19)–(24)] in zeroth order of $\rho_n/\rho\ll 1$, i.e., neglecting the contribution of thermal expansion, but at arbitrary values of the relative velocity w.

Using the general expression (19) for the phase velocity of second sound, one can calculate the longitudinal (x projection) and the transverse (y projection) components of the group velocity,

$$V_\parallel^{(\mathrm{gr})}=\frac{d\omega}{dk_\parallel}=V_d\pm\frac{V_\parallel^2k_\parallel}{\sqrt{V_\parallel^2k_\parallel^2+V_\perp^2k_\perp^2}}, \quad (28)$$

and

$$V_\perp^{(\mathrm{gr})}=\frac{d\omega}{dk_\perp}=\pm\frac{V_\perp^2k_\perp}{\sqrt{V_\parallel^2k_\parallel^2+V_\perp^2k_\perp^2}}, \quad (29)$$

where $k_\parallel=k\cos\theta$ and $k_\perp=k\sin\theta$. We see from Eqs. (19), (28), and (29) that, in general, the group speed is not equal to the phase speed. Furthermore the group velocity is not in the same direction as the phase velocity. Only for $\theta=0$ and $\theta=\pi$ are the phase and group velocities equal and in the same direction. The upper and lower signs in Eqs. (19), (28), and (29) correspond to the two possible waves of second sound. Eliminating the components of wave vector from Eqs. (28) and (29), we get the following equation:

$$\frac{(V_\parallel^{(\mathrm{gr})}-V_d)^2}{V_\parallel^2}+\frac{V_\perp^{(\mathrm{gr})2}}{V_\perp^2}=1. \quad (30)$$

Equation (30) gives the relation between the x component, $V_\parallel^{(\mathrm{gr})}$, and the y component, $V_\perp^{(\mathrm{gr})}$, of the group velocity of second sound. It illustrates the anisotropy of the dispersion relation (19). Equation (30) is the equation of an ellipse with the center at $(V_d,0)$ and half-axis $V_\parallel$ and $V_\perp$, in the plane with coordinates $(V_\parallel^{(\mathrm{gr})},V_\perp^{(\mathrm{gr})})$. We see that V_d has the features of a "drift" velocity, and $V_\parallel$ and $V_\perp$ have the features of longitudinal and transverse group velocities. At small values of the relative velocity (in the linear approximation with respect to w) this ellipse degenerates to a circle [see Eq. (27)] with a radius which is equal to the isotropic value of second-sound velocity under the condition $\rho_n/\rho\ll 1$. At high values of w there exist several possibilities.

In particular, the ellipse is strongly compressed in the longitudinal direction when the longitudinal $V_{\parallel}$ and transverse $V_{\perp}$ components are very different, $V_{\parallel} \ll V_{\perp}$. It is possible, when $V_{\parallel}^{(\mathrm{gr})}>0$, that the ellipse is situated completely in the right half plane from the ordinate axis. This case occurs when the drift velocity is greater than longitudinal velocity: $V_d > V_{\parallel}$. The positivity of the values $V_{\parallel}^2$ and $V_{\perp}^2$ in expressions (21) and (22) ensures that the condition of stability, for the solutions of the equations for superfluid helium, is satisfied. Below, we work in the region of stable solutions.

IV. RELATIONS BETWEEN THE OSCILLATING VALUES

We express the amplitudes of the oscillating variables of second sound as a function of the relative oscillation of entropy $\tilde{S}/S$, which is never zero. This is in contrast to the amplitudes of temperature $\tilde{T}$ and relative velocity $\tilde{w}_x$, which can be equal to zero under some conditions in anisotropic quasiparticle systems. To find the amplitudes, we substitute Eq. (6) into Eq. (17) and solve it together with Eq. (5) to find the amplitudes of temperature $\tilde{T}$ and relative velocity $\tilde{w}_x$ oscillations. In the zeroth order of ρ_n/ρ, where $\rho_n/\rho \ll 1$, we get

$$\tilde{T} = \frac{D}{E}\frac{\tilde{S}}{S}, \tag{31}$$

$$\tilde{w}_x = \frac{C}{E}\frac{S}{\rho_n}\frac{\tilde{S}}{S}, \tag{32}$$

where the coefficients

$$D = -\left(\frac{S}{\rho_n}\frac{\partial \ln \rho_n}{\partial w^2/2} - 2\frac{\partial \ln \rho_n}{\partial T}\right) w^2 \sin^2\theta + \frac{\partial \ln \rho_n}{\partial T}(u\cos\theta - w)w - \frac{S}{\rho_n}, \tag{33}$$

$$E = -\frac{S}{\rho_n}\frac{\partial \ln S}{\partial T}(1 + \Gamma w^2 \sin^2\theta), \tag{34}$$

$$C = -\left(2\frac{\partial \ln S}{\partial T} - \frac{\partial \ln \rho_n}{\partial T}\right) w \sin^2\theta - \frac{\partial \ln S}{\partial T}(u\cos\theta - w) \tag{35}$$

do not contain the small parameter ρ_n/ρ. In accordance with the schema used in this paper, we do not take into account further terms in the small parameter ρ_n/ρ in Eqs. (33)–(35).

From these expressions we see that oscillations of temperature and relative velocity are not zero in the zeroth order of the small parameter ρ_n/ρ in compliance with the assumptions made earlier. The amplitudes of oscillation of the chemical potential and superfluid velocity are first order in ρ_n/ρ [see Eq. (8) and Eq. (1), respectively]. Omitting the small term that includes $\tilde{\mu}$ in Eq. (12) and using Eqs. (31) and (32), we find the amplitude of the relative velocity projection onto the y axis,

$$\tilde{w}_y = \frac{\sin\theta}{u - w\cos\theta}\frac{S}{\rho_n}\frac{wC + D}{E}\frac{\tilde{S}}{S}. \tag{36}$$

From this equation it follows that if vector $\mathbf{k}$ is parallel to the relative velocity $\mathbf{w}$, then the value of $\tilde{w}_y$ is equal to zero because of the factor $\sin\theta$. This result satisfies the symmetry of the problem. Here we note that the projection of the relative velocity onto the y axis is zeroth order in the small parameter ρ_n/ρ.

To find the amplitude of the pressure oscillation in a second-sound wave in first order in the small parameter ρ_n/ρ, we use Eq. (15) and take into account the approximate equalities $\rho_s \approx \rho$ and $v_s \approx 0$ from Eq. (16). Then, using Eqs. (5)–(7), as well as Eqs. (31) and (32), we find the following expression:

$$\tilde{P} = \left(S(wC + D) + \rho_n u(w\cos\theta G - uE) + u^2\left[\frac{\partial\rho}{\partial T}D + \frac{\partial\rho}{\partial w^2/2}\frac{S}{\rho_n}wC\right]\right)\frac{1}{(1 - u^2/c^2)E}\frac{\tilde{S}}{S}, \tag{37}$$

where the coefficient G does not depend on the small parameter ρ_n/ρ,

$$G = -\frac{S}{\rho_n}\left(\frac{\partial \ln S}{\partial T}\right)\left[1 - \frac{\partial \ln \rho_n}{\partial \ln S} - \Gamma(u - w\cos\theta)w\cos\theta\right]. \tag{38}$$

It follows from expression (37) that the pressure oscillation is small in second sound and is first order in the small parameter ρ_n/ρ, as we supposed earlier.

The expression (8), together with Eqs. (31), (32), and (37), determines the chemical-potential $\tilde{\mu}$ oscillation, which from Eq. (1) directly gives the expression for the superfluid velocity oscillation amplitude,

$$\tilde{v}_{s\parallel} = \frac{1}{\rho u}\left(\tilde{P} - \frac{S(wC + D)}{E}\frac{\tilde{S}}{S}\right). \tag{39}$$

We next find expressions for the projection of the normal velocity oscillations onto the wave vector $\mathbf{k}$ and perpendicular to it,

$$\tilde{v}_{n\parallel} = (u - w\cos\theta)\frac{\tilde{S}}{S}, \tag{40}$$

$$\tilde{v}_{n\perp} = \tilde{w}_{\perp} = w_{\perp}\frac{G}{E}\frac{\tilde{S}}{S}. \tag{41}$$

So, Eqs. (31)–(41), together with the dispersion relation (19)–(24), solve the problem of the amplitude oscillation for the basic variables for second sound, in the approximation $\rho_n \ll \rho$, at arbitrary values of the relative velocity w. We see that it is mainly temperature and normal velocity which oscillate, while pressure and superfluid velocity only weakly oscillate. Thus, we have now verified all the assumptions we made earlier. Unlike the isotropic case when $w=0$, in the anisotropic case, the transverse component of normal velocity oscillations (41), (38), and (34) is not zero, i.e., the normal fluid has a component which oscillates in a direction perpendicular to the wave vector.

The relationships between the oscillating variables are much simplified when the wave vector **k** is perpendicular to **w**. Assuming $\theta=\pi/2$ in Eqs. (31)–(39) we get the following expressions for the oscillating variables in the two modes $u=\pm V_\perp$:

$$\tilde{w}_x = (V_d - w)\frac{\tilde{S}}{S}, \tag{42}$$

$$\tilde{w}_y = \pm V_\perp \frac{\tilde{S}}{S}, \tag{43}$$

$$\tilde{T} = \frac{\rho_n}{S}[V_\perp^2 - (V_d - w)w]\frac{\tilde{S}}{S}, \tag{44}$$

$$\tilde{P} = \frac{V_\perp^2}{1 - V_\perp^2/c^2}\left(\frac{\rho_n}{S}[V_\perp^2 - (V_d - w)w]\frac{\partial\rho}{\partial T} + (V_d - w)w\frac{\partial\rho}{\partial w^2/2}\right)\frac{\tilde{S}}{S}, \tag{45}$$

$$\tilde{v}_{s\parallel} = \pm\frac{1}{\rho}\frac{V_\perp}{1 - V_\perp^2/c^2}\left(\frac{\rho_n}{S}[V_\perp^2 - (V_d - w)w]\frac{\partial\rho}{\partial T} + (V_d - w)w\frac{\partial\rho}{\partial w^2/2} - \rho_n(1 - V_\perp^2/c^2)\right)\frac{\tilde{S}}{S}. \tag{46}$$

From Eqs. (42) and (43) it follows that in this "transverse" case, the transverse velocity $V_\perp$ completely determines the yth component of the amplitude of the relative velocity oscillation, and the difference $(V_d - w)$ determines its xth component. From Eq. (45), we see that the expression for the pressure oscillations, in this case, has two terms. The first is determined by the thermal-expansion coefficient, and the second is proportional to the square of w at small w.

For the "longitudinal" case, $\mathbf{k}\parallel\mathbf{w}$, we put $\theta=0$ in Eqs. (31)–(35). The relationships for the amplitudes in the two modes $u=V_d\pm V_\parallel$ are

$$\tilde{w}_x = (V_d - w \pm V_\parallel)\frac{\tilde{S}}{S}, \quad \tilde{w}_y = 0, \tag{47}$$

$$\tilde{T} = \frac{\rho_n}{S}\left[c_2^2 - \frac{\partial \ln \rho_n}{\partial \ln S}(V_d - w \pm V_\parallel)w\right]\frac{\tilde{S}}{S}. \tag{48}$$

From these expressions it follows that in the longitudinal case, unlike the transverse one, the two modes of second sound in general have different phase velocities $V_d \pm V_\parallel$ (which are equal to the group velocities) and different relationships between their amplitudes of the oscillating variables.

We now consider case when the relative velocity w is small and restrict ourselves to first order in its magnitude. From Eqs. (31)–(39) together with Eqs. (19), (26), and (27) we get

$$u = c_2 + w\cos\theta\left(2 - \frac{\partial \ln \rho_n}{\partial \ln S}\right), \tag{49}$$

$$\tilde{w}_x = \left[c_2\cos\theta + \left(1 - \frac{\partial \ln \rho_n}{\partial \ln S}\right)w\right]\frac{\tilde{S}}{S}, \tag{50}$$

$$\tilde{T} = \frac{\rho_n}{S}c_2^2\left[1 - \frac{w}{c_2}\cos\theta\frac{\partial \ln \rho_n}{\partial \ln S}\right]\frac{\tilde{S}}{S}, \tag{51}$$

$$\tilde{w}_y = c_2\sin\theta\frac{\tilde{S}}{S}. \tag{52}$$

For the oscillations of pressure and superfluid velocity, we obtain the following amplitudes:

$$\tilde{P} = \frac{c_2^2}{1 - c_2^2/c^2}\left\{c_2^2\frac{\rho_n}{S}\frac{\partial\rho}{\partial T} + \frac{w}{c_2}\cos\theta\left[-2\rho_n + c_2^2\frac{\partial\rho}{\partial w^2/2} + \frac{4c^2 - (3c^2 - c_2^2)\,\partial\ln\rho_n/\partial\ln S}{c^2 - c_2^2}c_2^2\frac{\rho_n}{S}\frac{\partial\rho}{\partial T}\right]\right\}\frac{\tilde{S}}{S}, \tag{53}$$

$$\tilde{v}_{s\parallel} = \frac{c_2}{\rho(1 - c_2^2/c^2)}\left\{c_2^2\frac{\rho_n}{S}\frac{\partial\rho}{\partial T} - \left(1 - \frac{c_2^2}{c^2}\right)\rho_n + \frac{w}{c_2}\cos\theta\left[-\left(1 + \frac{c_2^2}{c^2}\right)\rho_n + c_2^2\frac{\partial\rho}{\partial w^2/2} + 2\frac{c^2 + c_2^2 - c^2\,\partial\ln\rho_n/\partial\ln S}{c^2 - c_2^2}c_2^2\frac{\rho_n}{S}\frac{\partial\rho}{\partial T}\right]\right\}\frac{\tilde{S}}{S}. \tag{54}$$

From Eqs. (50) and (52) we see that the oscillations, which are transverse to the wave vector of the relative velocity, are first order in w,

$$\tilde{w}_\perp = w_\perp\left(1 - \frac{\partial \ln \rho_n}{\partial \ln S}\right)\frac{\tilde{S}}{S}. \tag{55}$$

From Eqs. (51)–(54) it follows that at $\theta=\pi/2$ the oscillations of temperature, pressure and superfluid velocity have no terms linear in w, in agreement with expressions (44)–(46). We note that the condition of applicability of these expressions is that w is small in comparison with the velocity of second sound c_2.

At $w=0$, i.e., in the isotropic case, Eqs. (50)–(54) give the results of Ref. 11 (for the case $\rho_n \ll \rho$), when the first term in the right-hand side (rhs) of Eq. (54), which includes the coefficient of thermal expansion, can be neglected compared to the second term, which contains the normal density ρ_n. In contrast when phonons dominate, both terms contribute the same order of magnitude. This follows from the thermodynamic relation (9) together with the phonon equation of state.

V. SECOND SOUND IN ANISOTROPIC PHONON SYSTEMS

The general relations obtained above can be much simplified when the normal fluid is comprised only of phonons. These systems are presently most interesting from the point of view of experiments,[6,9,10] where strongly anisotropic systems (phonon pulses) with condition $\rho_n \ll \rho$ are created. These phonon systems are characterized by large values of the relative velocity w, which is close to the Landau critical velocity when there are only phonons. There is a direct evidence for this: a phonon pulse propagates in superfluid helium as a whole, with a velocity which is experimentally indistinguishable from the Landau critical velocity for phonons. For phonons with a linear energy-momentum relation, $\varepsilon(p)=cp$, the Landau critical velocity is equal to c.[2]

It is well known that in stationary conditions, it is impossible to create a superfluid state in helium with a relative velocity close to the Landau critical velocity. However in phonon pulses with duration about $t_p = 1\times 10^{-5} - 1\times 10^{-7}$ s, it is possible to create large values of the relative velocity $w \approx 0.98c$ without superfluidity breaking down. On the other hand, in experiments[6,9,10] the phonon pulse duration is much greater than the time to attain equilibrium, mainly due to interactions by three phonon processes, with characteristic time $t_{3pp} \sim 1\times 10^{-8}$ s for the same experimental conditions.[6,9,10] Fast relaxation has been observed directly in the experiments with colliding phonon pulses,[21] in which the phenomenon of the "hot-line" appears (see also Ref. 22). Thus we can use a hydrodynamic approach for describing the second sound in phonon pulses if the period τ of the second-sound wave satisfies the inequalities $t_{3pp} \ll \tau \ll t_p$.

For phonons with a linear energy-momentum relation, the normal-fluid density and entropy are determined by the equations in Ref. 11,

$$\rho_n = \frac{2\pi^2}{45}\frac{T^4}{\hbar^3 c^5 (1-\bar{w}^2)^3}, \tag{56}$$

$$\frac{S}{\rho_n} = \frac{c^2 - w^2}{T}, \tag{57}$$

where we denote $\bar{w} = w/c$.

Using Eqs. (20)–(24) taking into account Eqs. (56) and (57) we find the following expressions for the drift velocity V_d, longitudinal $V_\parallel$, and transverse $V_\perp$ velocities:

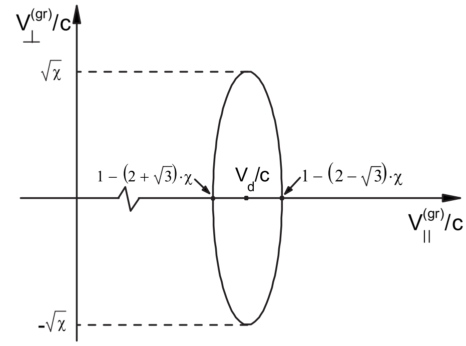

FIG. 2. The relation between the longitudinal $V_\parallel^{(gr)}/c$ and transverse $V_\perp^{(gr)}/c$ components of second-sound group velocity for a strongly anisotropic phonon system. ($\chi = 1 - w/c, \chi \ll 1$).

$$V_d = \frac{2}{3}\frac{w}{1-\bar{w}^2/3}, \tag{58}$$

$$V_\parallel = \frac{c}{\sqrt{3}}\frac{1-\bar{w}^2}{1-\bar{w}^2/3}, \tag{59}$$

$$V_\perp = \frac{c}{\sqrt{3}}\sqrt{\frac{1-\bar{w}^2}{1-\bar{w}^2/3}}. \tag{60}$$

From Eqs. (59) and (60) we find that $w<c$ for $V_\parallel$ and $V_\perp$ to be real. This coincides with the condition for thermodynamic stability for systems with a linear dispersion law (see Ref. 2), and also with the Landau criterion that determines the critical velocity in quasiparticle systems.

From Eqs. (58)–(60) we see that the drift velocity V_d monotonically grows, with increasing w, from zero at $w=0$ to $V_d=c$ at $w=c$. At small values of $w \ll c$, the drift velocity $V_d = 2w/3$. The longitudinal $V_\parallel$ and the transverse $V_\perp$ velocities monotonically decrease from their maximum values at $w=0$, which is equal to the isotropic second-sound velocity $c/\sqrt{3}$ in the limit $\rho_n \ll \rho$, to their minimum value which is zero at $w=c$. It should be noted that for phonon systems, the transverse velocity is always greater than the longitudinal velocity $V_\perp \geq V_\parallel$. At $w > c/\sqrt{3}$, the value $V_d > V_\parallel$. So, the ellipse is completely in the right half plane ($V_x^{(gr)} > 0$).

In the limiting case of strongly anisotropic systems, when $1 - w/c = \chi \ll 1$ from Eqs. (58)–(60) it follows:

$$V_d = c(1-2\chi), \quad V_\parallel = c\sqrt{3}\chi, \quad V_\perp = c\sqrt{\chi}. \tag{61}$$

In this case, the diagram of group velocity, calculated for the typical experimental value of $\chi = 0.02$, has the ratio between its half axis $V_\perp / V_\parallel = 1/\sqrt{3\chi} = 4$. This is shown in Fig. 2 for a second-sound mode in a strongly anisotropic system $\chi \ll 1$. In Fig. 2, the central point V_d moves with velocity $V_d = c(1 - 2\chi) \leq c$, which is close to the velocity of first sound.

Second-sound dispersion relation (19) for phonon systems, taking account Eqs. (58)–(60), can be written as follows:

$$\frac{u}{c} = \frac{2\cos\theta\bar{w} + R}{3 - \bar{w}^2}, \tag{62}$$

where we denote

$$R = \pm\sqrt{(1-\bar{w}^2)(3-(2\cos^2\theta+1)\bar{w}^2)}. \tag{63}$$

When $\theta=0$ we see from the general Eqs. (19), (28), and (29) that the phase velocity is equal to the group velocity. We calculate the velocity for the phonon system from Eqs. (62) and (63) with $\theta=0$. The two velocities u_+ and u_-, which correspond to + and − signs in Eq. (63), respectively, are shown in Fig. 3. The velocity u_- occurs when the second-sound wave propagates in the opposite direction to the normal fluid. We see that u_- changes from negative to positive as w increases. When $u_- < 0$, the second-sound wave propagates to the left faster than $\mathbf{w}$ propagates to the right, and viceversa when $u_- > 0$. At $w/c=0.58$ the second-sound wave is stationary in the laboratory frame. This condition is connected to wave blocking in classical systems (see, e.g., Ref. 17 and horizons in white hole analogs[18,19]).

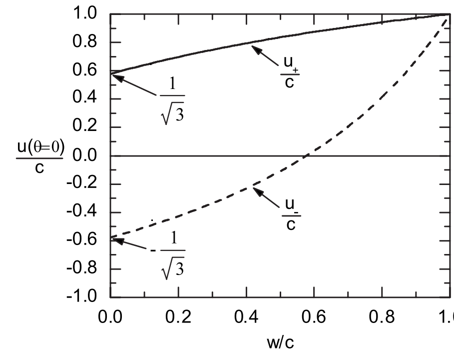

FIG. 3. The velocities u_+/c and u_-/c are shown as a function of w/c when $\mathbf{w}$ and $\mathbf{k}$ are parallel, i.e., $\theta=0$, for the normal fluid comprised only of phonons. The group and phase velocities are equal when $\theta=0$. The velocities u_+/c and u_-/c are for the second-sound wave propagating parallel and antiparallel to the velocity of the normal fluid, respectively. When $w/c=1/\sqrt{3}$, $u_-/c=0$ and the second-sound wave is stationary in the laboratory frame.

VI. AMPLITUDE RELATIONS IN ANISOTROPIC PHONON SYSTEMS

From the general relations (31)–(36), which express the amplitudes of the oscillating variables, in second sound, in zeroth order of $\rho_n/\rho \ll 1$, we obtain using the phonon equations of state (56) and (57)

$$\frac{\tilde{T}}{T} = -\frac{(2\cos^2\theta+1)\bar{w}^4 - 2\sin^2\theta\bar{w}^2 + 4R\cos\theta\bar{w} - 3}{(3-\bar{w}^2)[3-(2\cos^2\theta+1)\bar{w}^2]}\frac{\tilde{S}}{S}, \tag{64}$$

$$\frac{\tilde{w}_x}{c} = \frac{(1-\bar{w}^2)[(2\cos^2\theta+1)\bar{w}^3 - 3\bar{w} + 3R\cos\theta]}{(3-\bar{w}^2)[3-(2\cos^2\theta+1)\bar{w}^2]}\frac{\tilde{S}}{S}, \tag{65}$$

$$\frac{\tilde{w}_y}{c} = -\sin\theta\frac{(1-\bar{w}^2)[(2\cos^2\theta+1)\bar{w}^2 + R\cos\theta\bar{w} - 3]}{[(\bar{w}^2-1)\bar{w}\cos\theta + R][3-(2\cos^2\theta+1)\bar{w}^2]}\frac{\tilde{S}}{S}. \tag{66}$$

The transverse component of the relative velocity of a second-sound wave in phonon system, from Eqs. (41) and (38) taking into account Eqs. (56) and (57), is

$$\frac{\tilde{w}_\perp}{w_\perp} = -\frac{(2\cos^2\theta+1)\bar{w}^4 - 2(2+\cos^2\theta)\bar{w}^2 + 2R\cos\theta\bar{w} + 3}{(3-\bar{w}^2)[3-(2\cos^2\theta+1)\bar{w}^2]}\frac{\tilde{S}}{S}. \tag{67}$$

The components of the relative velocity $\mathbf{w}$ given by the relations (65)–(67) also give the components of the normal-fluid velocity, in the same zeroth-order approximation of the small value $\rho_n/\rho \ll 1$, because the superfluid velocity in the wave of second sound is much smaller, of first order in $\rho_n/\rho \ll 1$. This follows from the general expressions (37)–(39).

The derivatives of density with respect to temperature and relative velocity, occur in the general Eqs. (37) and (39) for oscillations of pressure and superfluid velocity. Using the thermodynamic relations (9) and (11), we can express them by the derivatives of entropy and normal density with respect to pressure. Substituting relations (56) and (57) into Eqs. (9) and (11), we obtain for the phonon system

$$\frac{\partial\rho}{\partial T} = -\frac{S[3u_G + 1 + (u_G - 1)\bar{w}^2]}{c^2(1-\bar{w}^2)}, \tag{68}$$

$$\frac{\partial \rho}{\partial w^2/2} = -\frac{\rho_n[5u_G + 1 + (u_G - 1)\bar{w}^2]}{c^2(1-\bar{w}^2)}, \tag{69}$$

where $u_G = (\rho/c)(\partial c/\partial \rho) = 2.84$ is the Gruneisen constant.

Expressions for the pressure and superfluid velocity in a second-sound wave in a phonon system can be found, to a first approximation with respect to the small value ρ_n/ρ, by substituting Eqs. (68) and (69) into the general relations (37) and (39) taking account of Eqs. (33)–(35) and (38). As a result we find

$$\tilde{P} = \rho_n c^2\{-4[(u_G-2)\bar{w}^4 - 2(3u_G+1)\bar{w}^2 + 9u_G + 8]\bar{w}^4 \cos^4\theta - 2R[u_G\bar{w}^4 - 2(3u_G+5)\bar{w}^2 + 9u_G + 14]\bar{w}^3\cos^3\theta - 2(3-\bar{w}^2)(3\bar{w}^4 + 2u_G\bar{w}^2 - 6u_G - 7)\bar{w}^2\cos^2\theta + (3-\bar{w}^2)R[(u_G+1)\bar{w}^4 - 2(4u_G+7)\bar{w}^2 + 15u_G + 21]\bar{w}\cos\theta - (1-\bar{w}^2) \times (3-\bar{w}^2)^2[(u_G+1)\bar{w}^2 - 3u_G - 1]\}\{2(3-\bar{w}^2)[3-(2\cos^2\theta+1)\bar{w}^2][\bar{w}^4\cos^2\theta + \bar{w}^2\cos^2\theta + \bar{w}^2 + 2R\bar{w}\cos\theta - 3]\}^{-1}\frac{\tilde{S}}{S}, \tag{70}$$

$$\frac{\tilde{v}_{s\parallel}}{c} = -(u_G+1)\frac{\rho_n}{2\rho}(3-\bar{w}^2)^2[4\bar{w}^4\cos^4\theta + 2\bar{w}^3\cos^3\theta R - 4\bar{w}^2\cos^2\theta - (5-\bar{w}^2)R\bar{w}\cos\theta - \bar{w}^4 + 4\bar{w}^2 - 3] \times \{[3-(2\cos^2\theta+1)\bar{w}^2]\}^{-1}\{(6\bar{w}^2-2)\bar{w}^3\cos^3\theta + (\bar{w}^2+5)R\bar{w}^2\cos^2\theta + (\bar{w}^2-3)(2\bar{w}^3\cos\theta + R)\}^{-1}\frac{\tilde{S}}{S}. \tag{71}$$

Thus Eqs. (64)–(66), (70), and (71), using (63), determine explicitly the oscillations of temperature, the relative velocity components, pressure and superfluid velocity, in a second-sound wave in an anisotropic phonon system. These are expressed in terms of the oscillations of entropy, in the approximation of the first nonvanishing terms containing the small ratio ρ_n/ρ. Note that the dispersion law, for the two modes of second sound in anisotropic phonon systems, is given by Eqs. (62) and (63). When w is not small compared to c, we see, from Eqs. (64)–(66), (70), (71), and (63), that the amplitude relations depend strongly on the angle θ between the wave vector **k** and the relative velocity **w**.

In the linear approximation with respect to $w \ll c$, the relations (64)–(66), (70), and (71), taking into account Eq. (63), are simplified,

$$\frac{\tilde{T}}{T} = \left(\frac{1}{3} \mp \frac{4}{3\sqrt{3}}\cos\theta\bar{w}\right)\frac{\tilde{S}}{S}, \tag{72}$$

$$\frac{\tilde{w}_x}{c} = \left(\pm\frac{1}{\sqrt{3}}\cos\theta - \frac{1}{3}\bar{w}\right)\frac{\tilde{S}}{S}, \tag{73}$$

$$\frac{\tilde{w}_y}{c} = \pm\frac{1}{\sqrt{3}}\sin\theta\frac{\tilde{S}}{S}, \tag{74}$$

$$\tilde{P} = -\rho_n c^2\left(\frac{3u_G+1}{2} \pm \frac{\sqrt{3}}{18}(21u_G+23)\cos\theta\bar{w}\right)\frac{\tilde{S}}{S}, \tag{75}$$

$$\frac{\tilde{v}_{s\parallel}}{c} = \mp\frac{\rho_n}{\rho}(u_G+1)\left(\frac{\sqrt{3}}{2} \pm \frac{5}{2}\cos\theta\bar{w}\right)\frac{\tilde{S}}{S}. \tag{76}$$

In Eqs. (72)–(76) the upper and lower signs correspond to the ones in the expression

$$u = \pm\frac{c}{\sqrt{3}} + \frac{2}{3}\cos\theta w \tag{77}$$

for the dispersion law for two modes of second sound in a phonon system in the same linear approximation where $w \ll c$.

In Figs. 4–8 we illustrate the amplitude relationships for

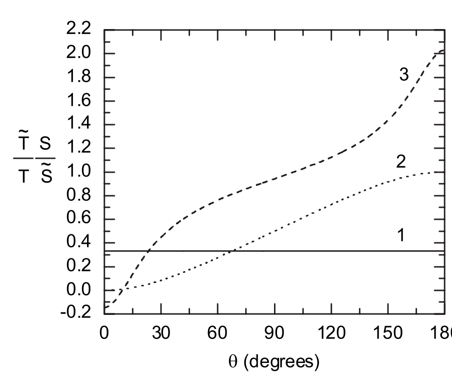

FIG. 4. The ratio of the relative amplitudes $\tilde{T}S/(T\tilde{S})$ for the second-sound mode as a function of the angle θ for phonon systems at $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eq. (64) with account of Eq. (63) with sign "plus."

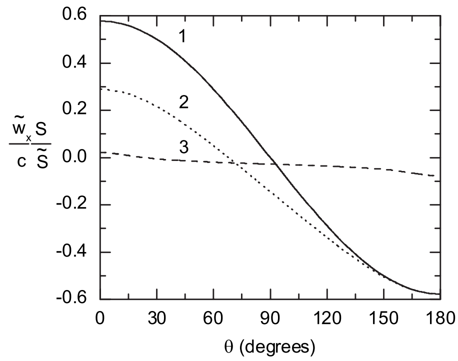

FIG. 5. The angular dependency of ratio $\tilde{w}_x S/(c\tilde{S})$ for second-sound mode for phonon systems at $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eqs. (65) and (63) with sign "plus."

the second-sound mode with "plus" sign in Eq. (63), which occurs in Eqs. (64)–(66), (70), and (71).

In Fig. 4 we show the ratio of the relative amplitudes $\tilde{T}S/(T\tilde{S})$ for the second-sound mode as a function of the angle θ, for phonon systems at $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eq. (64) taking into account Eq. (63). For the case $w=0$ (curve 1) this dependence is isotropic in accordance with Eq. (77). For $\bar{w}=1/\sqrt{3}$, which is equal to the isotropic phonon second-sound velocity, the temperature oscillations are zero at $\theta=0$ (curve 2). At $\bar{w}>1/\sqrt{3}$ the temperature oscillation becomes a sign-changing function; it increases monotonically from a negative value at $\theta=0$ to a positive value at $\theta=\pi$ (curve 3). Thus, in anisotropic phonon systems in second-sound mode, it is possible that the amplitude of temperature oscillations is zero.

The angular dependences of ratios $\tilde{w}_x S/(c\tilde{S})$ and $\tilde{w}_y S/(c\tilde{S})$ are shown in Figs. 5 and 6. They are calculated from Eqs. (65), (66), and (63). Curve 1 corresponds to phonon systems with $w=0$, curve 2 corresponds to $\bar{w}=1/\sqrt{3}$, and curve 3 corresponds to $\bar{w}=0.97$. We see that the amplitude of the oscillation of the relative velocity decreases in absolute value as the relative velocity increases.

In Figs. 7 and 8 we show the angular dependences of the ratios $\tilde{v}_{s\parallel}\rho S/(c\rho_n\tilde{S})$ and $\tilde{P}S/(\rho_n c^2\tilde{S})$ corresponding to phonon systems with $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eqs. (70) and (71) taking into account Eq. (63). We see that the amplitudes of oscillations of superfluid velocity and pressure in a phonon second-sound wave increases by two orders of magnitude at small angles, when the relative velocity increases from value $w=0$ to $\bar{w}=0.97$. It should be noted that for small oscillations, which are under consideration here, $\tilde{S}/S \ll 1$. Therefore, taking into account the strong inequality $\rho_n/\rho \ll 1$, the relative amplitudes of superfluid velocity and pressure in second sound are very small.

Figures 4–8 correspond to the second-sound mode with the "plus" sign in Eq. (63), which enters into formulas (64)–(66), (70), and (71). The second-sound mode, corresponding to the "minus" sign in Eq. (63), has amplitude relations which can be obtained from the corresponding relations for the plus sign second-sound mode by substituting $\theta \to \pi-\theta$, $\tilde{w}_y \to -\tilde{w}_y$, $\tilde{v}_{s\parallel} \to -\tilde{v}_{s\parallel}$. It follows from relations (64)–(66), (70), and (71) that the other values are unchanged. Thus we can see from Fig. 5 that for strongly anisotropic phonon system (curve 3), the minus sign second-sound

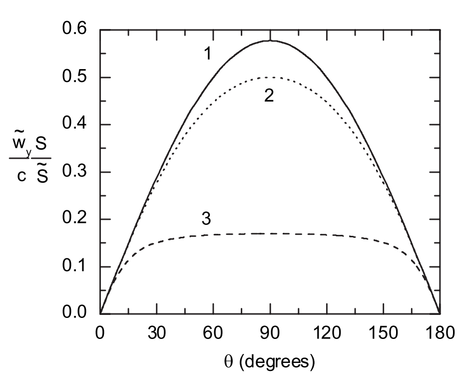

FIG. 6. The angular dependency of ratio $\tilde{w}_y S/(c\tilde{S})$ for second-sound mode for phonon systems at $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eqs. (66) and (63) with "plus" sign.

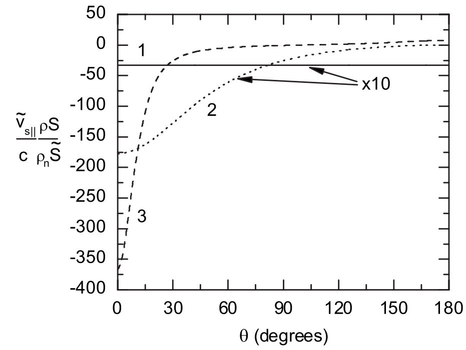

FIG. 7. The ratio of the relative amplitudes $\tilde{v}_{s\parallel}\rho S/(c\rho_n\tilde{S})$ for the second-sound mode as a function of the angle θ for phonon systems at $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eqs. (71) and (63) with "plus" sign. The ratios for the cases $w=0$ (curve 1) and $\bar{w}=1/\sqrt{3}$ (curve 2) are shown multiplied by a factor 10.

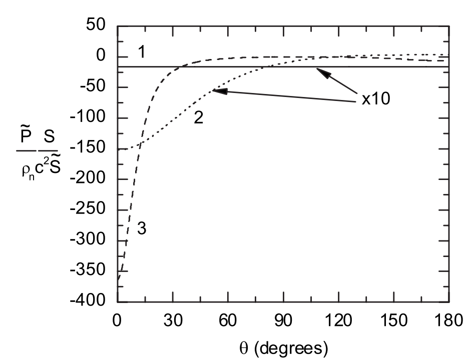

FIG. 8. The ratio of the relative amplitudes $\tilde{P}S/(\rho_n c^2\tilde{S})$ for the second-sound mode as a function of the angle θ for phonon systems at $w=0$ (curve 1), $\bar{w}=1/\sqrt{3}$ (curve 2), and $\bar{w}=0.97$ (curve 3), calculated from Eqs. (70) and (63) with "plus" sign. The ratios for the cases $w=0$ (curve 1) and $\bar{w}=1/\sqrt{3}$ (curve 2) are shown multiplied by a factor 10.

mode, bearing in mind the transformation $\theta \rightarrow \pi - \theta$, has relative temperature oscillations, whose amplitude is not small at small angles.

VII. CONCLUSION

The main feature of a strongly anisotropic phonon system in superfluid helium, created in experiments,[1,6,9,10] is the high value of the relative velocity, w, between the superfluid and normal components. The second-sound mode in stationary ($w=0$) helium and when w is small has been studied for many years, but the analysis of second-sound propagation at arbitrary w has not been done until now.

In this paper, the dispersion relation for second-sound modes of superfluid ^{4}He is obtained for arbitrary values of the relative velocity w, when the thermal excitation contribution is small, i.e., $\rho_n/\rho \ll 1$. We show that the phase and group velocities of second sound, in general, depend strongly on the angle between the wave vector and the relative velocity of the normal and superfluid components, and can be characterized by three characteristic velocities, the longitudinal $V_{\parallel}$, perpendicular $V_{\perp}$, and drift V_d velocities. The second-sound group-velocity diagram is an ellipse in the plane with axes $V_{\perp}^{(gr)}$ and $V_{\parallel}^{(gr)}$. We have found the relationships between the amplitudes of the oscillating variables for second sound. In the general case, the normal fluid not only has a velocity component parallel to the wave vector, but also one transverse to the wave vector. It is shown that mainly temperature and the normal-fluid velocity oscillate, whereas the oscillations of pressure and the superfluid velocity are small. In the limiting case $w=0$, the general relations for the second-sound amplitudes obtained here are the same as those in Ref. 11 when $\rho_n/\rho \ll 1$, which is the condition considered in this paper.

The velocities and the relationships between the amplitudes of the oscillating variables, in a second-sound wave, are studied in detail for the case of an anisotropic phonon system with arbitrary w. This condition is very important in practice because high values of w are realized in phonon pulses propagating in superfluid helium.[1,6,9,10] For this case we find that the amplitude of the temperature oscillation in a second-sound wave in an anisotropic phonon system can be zero at small angles between the relative velocity and the wave vector (see Fig. 4). The amplitudes of oscillations of the superfluid velocity and pressure, in a phonon second-sound wave, increase by two orders of magnitude, in the region of small angles (see Figs. 7 and 8), when the relative velocity increases from $w=0$ to $w/c=0.97$. The latter value is close to the typical experimental values in Refs. 9 and 10.

So, the second-sound mode in superfluid helium, at nonzero values of the relative motion w, has unusual properties. They are most apparent at large values of w. The authors hope that the expressions for the phase and group velocities, and for the relationships between the amplitudes of the oscillating variables in the second-sound mode of superfluid helium, when there is relative motion between the normal fluid and superfluid, obtained in this paper, will stimulate new experiments to study second-sound propagation in anisotropic quasiparticle systems of superfluid helium.

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